MATH20132 Calculus of Several Variables. 2020-21

Problems 7: Graphs, level sets, parametric sets, Implicit & Inverse functions

Inverses

1. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \exp(x)\cos(y) \\ \exp(x)\sin(y) \end{pmatrix},$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$.

i. Prove that the linear map $d\mathbf{f}_{\mathbf{a}} : \mathbb{R}^2 \to \mathbb{R}^2$ has an inverse for all $\mathbf{a} \in \mathbb{R}^2$ but that $\mathbf{f}$ does not have an inverse.

ii. Let $\mathbf{f}_0 : U = \{ (x, y)^T \in \mathbb{R}^2 \mid -\pi < y < \pi \} \to \mathbb{R}^2$ be the restriction of $\mathbf{f}$. Prove that $\mathbf{f}_0$ is an injection.

iii. If $\mathbf{g} : \mathbf{f}_0(U) \to U$ is inverse of $\mathbf{f}_0$ find the Jacobian matrix of $\mathbf{g}$ at $\mathbf{b} = \mathbf{f}_0(\mathbf{a})$.

Hint: Use the Chain Rule.

2. State Inverse Function Theorem:

i. Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{x} \mapsto (x^2y, xy^2)^T$.

a. Prove that $\mathbf{f}$ has a local inverse at $\mathbf{a} = (a, b)^T$ (i.e. has an differentiable inverse when restricted to some open set containing $\mathbf{a}$) if and only if $ab \neq 0$.

b. Find the Jacobian matrix $J\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big)$ of the local inverse $\mathbf{g} = \mathbf{f}^{-1}$ at $\mathbf{f}(\mathbf{a})$, when it exists.

ii. Define the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{x} \mapsto (yz, xz, xy)^T$.

a. Prove that $\mathbf{f}$ has a local inverse at $\mathbf{a} = (a, b, c)^T$ (i.e. has an differentiable inverse when restricted to an open set containing $\mathbf{a}$) if and only if $abc \neq 0$.

b. Find the Jacobian matrix $J\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big)$ of the local inverse $\mathbf{g} = \mathbf{f}^{-1}$ at $\mathbf{f}(\mathbf{a})$, when it exists.

3. **Proof of the Inverse Function Theorem** Assume that $\mathbf{f} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function such that at $\mathbf{a} \in U$ the Jacobian matrix $J\mathbf{f}\,(\mathbf{a})$ is of full-rank. Prove that the Inverse Function Theorem follows by applying the Implicit Function Theorem to the function $\mathbf{h} : \mathbb{R}^n \times U \subseteq \mathbb{R}^{2n} \to \mathbb{R}^n$ defined by

$$\mathbf{h}\left(\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right) = \mathbf{x} - \mathbf{f}(\mathbf{y}), \quad \text{where} \quad \mathbf{x} \in \mathbb{R}^n,\, \mathbf{y} \in U.$$

Hint: The important observation is that by definition of $\mathbf{h}$,

$$\mathbf{h}\left(\begin{pmatrix}\mathbf{f}(\mathbf{a})\\ \mathbf{a}\end{pmatrix}\right) = \mathbf{f}(\mathbf{a}) - \mathbf{f}(\mathbf{a}) = \mathbf{0}.$$

So, setting

$$\mathbf{p} = \begin{pmatrix}\mathbf{f}(\mathbf{a})\\ \mathbf{a}\end{pmatrix},$$

we have $\mathbf{h}(\mathbf{p}) = \mathbf{0}$ as required for the Implicit Function Theorem. To deduce anything from that Theorem it is required that $J\mathbf{h}\,(\mathbf{p})$ is of full rank. What is $J\mathbf{h}\,(\mathbf{p})$?

Image or Parametric sets are locally graphs.

4. Show that the following Image sets are locally graphs around the point given.

i. $\left\{((4+2\cos t)\cos s, (4+2\cos t)\sin s, 2\sin t)^T : s,t \in \mathbb{R}\right\}$ with $\mathbf{q} = (3\pi/4, \pi/4)^T$,

ii. $\left\{(xy^2,\, x^2+y,\, x^3-y^2,\, y^2)^T : x,y \in \mathbb{R}\right\}$, with $\mathbf{q} = (-1,2)^T$,

iii. $\left\{(\cos t,\, \sin t,\, t)^T : t \in \mathbb{R}\right\}$ with $q = 3\pi$.

Higher order derivatives

5. Return to Question 7 on Sheet 6. We showed that for the level set of $(x, y, u, v)^T \in \mathbb{R}^4$ satisfying

$$\begin{aligned} x^2 + y^2 + 2uv &= 4 \\ x^3 + y^3 + u^3 - v^3 &= 0, \end{aligned}$$

there exists an open subset of $\mathbb{R}^4$ containing the solution $\mathbf{p} = (-1, 1, 1, 1)$ in which the u and v can be given as functions of x and y, with $(x, y)^T$ in some open subset of $\mathbb{R}^2$ containing the point $\mathbf{q} = (-1, 1)^T$. Find the second order derivatives of u and v.

A purpose of Question 7 was to highlight the fact that when conditions are satisfied the Implicit Function Theorem ensures that functions exist, but gives no further information about them. Nonetheless their derivatives can be found. In this question we continue to find their second derivatives.

Advice for Exams *Know how to take second derivatives. Many students failed to calculate correctly second derivatives in the exam. Make sure that the exam is not the first time you attempt a question such as this one.*

Best Affine Approximations.

Recall that the Best Affine Approximation to a function $\mathbf{f}$ *at a point* $\mathbf{a}$ *is given by*

$$\mathbf{f}(\mathbf{a}) + d\mathbf{f}_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) = \mathbf{f}(\mathbf{a}) + J\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}) .$$

6. Write down the Best Affine Approximation to

i. $f(\mathbf{x}) = x(x + y)$ at $\mathbf{a} = (2, -1)^T$, and what value does the approximation give at $\mathbf{a}' = (2.1, -0.9)^T$?

ii. $f(\mathbf{x}) = xy + yz + xz$ at $\mathbf{a} = (-1, -1, 4)^T$, and what value does the approximation give at $\mathbf{a}' = (-0.9, -1.1, 4.1)^T$?

iii.
$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} xy^2 \\ x^2 y \end{pmatrix}$$

at $\mathbf{a} = (2, -3)^T$, and what value does the approximation give at $\mathbf{a}' = (1.9, -3.1)^T$?

Hint These functions have been seen previously on Sheet 4.

7. Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{f}(\mathbf{x}) = (x^3 - 2xy^2,\ x+y)^T$. Show that $\mathbf{f}$ locally invertible at $\mathbf{a} = (1, -1)^T$.

What is the Best Affine Approximation to the **inverse** function near $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (-1,\ 0)^T$?

What approximation does this give to $\mathbf{f}^{-1}\left((-0.9,\ 0.1)^T\right)$?

Tangent Spaces for Graphs

8. For each of the following scalar-valued functions ϕ, find both a basis for the Tangent Space and the equation of the Tangent Plane to the graph of φ for the given point on the graph. For the latter give your answer in the form "the Tangent plane to φ at $\mathbf{q}$ is the graph of the function $g(\mathbf{x}) = \ldots$".

i. $\varphi(\mathbf{x}) = 4x^2 + y^2, \qquad \mathbf{q} = (1, -1)^T \in \mathbb{R}^2,$

ii. $\varphi(\mathbf{x}) = \sqrt{9 - x^2 - y^2}, \qquad \mathbf{q} = (2, 1)^T \in \mathbb{R}^2,$

iii. $\varphi(\mathbf{x}) = 9 - x^2 - y^2, \qquad \mathbf{p} = (2, -2, 1)^T \in G_\varphi,$

iv. $\varphi(\mathbf{x}) = 5/\left(1 + x^2 + 3y^2\right), \qquad \mathbf{p} = (1, -1, 1)^T \in G_\varphi.$

9. Repeat Question 8 for the **vector-valued** function

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{pmatrix} xy \\ x^2 + y^2 \end{pmatrix}$$

with $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, at $\mathbf{p} = (2, -1, -2, 5)^T \in G_{\boldsymbol{\phi}}$.

Additional Questions 7

9. Explain why

$$\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} xy \\ x^2-y^2 \end{pmatrix}$$

is invertible in some neighbourhood of $\mathbf{p}=(1,1)^T$.

Calculate the Fréchet derivative of the inverse at $\mathbf{q}=\mathbf{f}(\mathbf{p})=(1,0)^T$.

10. At what points are the functions below, from $\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$ to $\mathbb{R}^3$, invertible?

i.

$$\mathbf{f}_1(\mathbf{x})=\left(x^2+2xz,\, 2\sqrt{y},\, z^2+xz\right)^T,$$

ii

$$\mathbf{f}_2(\mathbf{x})=\left(x^2+xz,\, 2\sqrt{y},\, z^2+xz\right)^T,$$

iii.

$$\mathbf{f}_3(\mathbf{x})=\left(x^2+xz/2,\, 2\sqrt{y},\, z^2+xz\right)^T.$$

11. Consider the surface in $\mathbb{R}^4$:

$$\left\{\begin{pmatrix} yz \\ xz \\ xy \\ xyz \end{pmatrix} : \mathbf{x}\in\mathbb{R}^3\setminus\{\mathbf{0}\}\right\} = \left\{\begin{pmatrix} \mathbf{f}(\mathbf{x}) \\ xyz \end{pmatrix} : \mathbf{x}\in\mathbb{R}^3\setminus\{\mathbf{0}\}\right\},$$

where $\mathbf{f}(\mathbf{x})=(yz,xz,xy)^T$. The point $\mathbf{p}=(-2,2,-1,-2)^T$ on the surface is the image of $\mathbf{a}=(1,-1,2)^T$.

I have introduced the function $\mathbf{f}$ since it was the subject of Question 2ii where it was shown that $\mathbf{f}$ has an inverse in an open set containing $\mathbf{a}$. That is there exists $V:\mathbf{a}\in V\subseteq\mathbb{R}^3$ and $W\subseteq\mathbb{R}^3$ such that $\mathbf{f}:V\to W$ has an inverse, $\mathbf{g}$ say. For a general $\mathbf{x}\in V$ write $\mathbf{s}=\mathbf{f}(\mathbf{x})$, so $\mathbf{s}\in W\subseteq\mathbb{R}^3$. Also, let $\mathbf{b}=\mathbf{f}(\mathbf{a})=(-2,2,-1)^T$.

Since $\mathbf{g}$ is the inverse of $\mathbf{f}$ we have $\mathbf{x}=\mathbf{g}(\mathbf{s})$. This means the coordinates of $\mathbf{x}$, i.e. x,y and z, can be written as functions of $\mathbf{s}$. Thus xyz is a function of $\mathbf{s}$, i.e. $xyz=\phi(\mathbf{s})$, say. Then our surface is locally a graph:

$$\left\{\begin{pmatrix} \mathbf{f}(\mathbf{x}) \\ xyz \end{pmatrix} : \mathbf{x}\in V\right\} = \left\{\begin{pmatrix} \mathbf{s} \\ \phi(\mathbf{s}) \end{pmatrix} : \mathbf{s}\in W\right\}. \tag{1}$$

We know little about $\phi : W \to \mathbb{R}$, though you can check that $\phi(\mathbf{b}) = -2$. We can, though, calculate the derivative at $\mathbf{b}$.

Question Calculate $J\phi(\mathbf{b})$.

Hint Write $k(\mathbf{x}) = xyz$ for $\mathbf{x} \in V$, express ϕ as a convolution of $\mathbf{g}$ and k and apply the Chain Rule .

12. Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{f}\left((u,v)^T\right) = (u^3 + uv + v^3,\ u^2 - v^2)^T$. Show that $\mathbf{f}$ locally invertible at $\mathbf{a} = (1,1)^T$.

What is the Best Affine Approximation to the **inverse** function near $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (3,0)^T$?

What approximation does this give to $\mathbf{f}^{-1}(\mathbf{b}')$ where $\mathbf{b}' = (3.1, -0.2)^T$?

13. *For additional practice* For each function, find the Best Affine Approximation at the given point. With $\mathbf{s} = (s,t)^T \in \mathbb{R}^2$,

i. $\mathbf{f}(\mathbf{s}) = (t\cos s, t\sin s, t)^T$, $\mathbf{q} = (\pi/2, 2)^T$,

ii. $\mathbf{f}(\mathbf{s}) = (t^2\cos s, t^2, t^2\sin s)^T, \mathbf{q} = (0,1)^T$,

14 In each of the following examples, find both a basis for the Tangent Space and the equation of the Tangent Plane to the graph of ϕ for the given point:

i. $\phi(\mathbf{x}) = x(x+y)$, at $\mathbf{q} = (2,-1)^T \in \mathbb{R}^2$,

ii. $\phi(\mathbf{x}) = (x-1)^2 + y^2$ at $\mathbf{q} = (0,2)^T \in \mathbb{R}^2$,

iii. $\phi(\mathbf{x}) = \sin(xy^2z^3)$ at $\mathbf{q} = (\pi, 1, -1)^T$.

iv.

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}$$

at $\mathbf{q} = (2,-3)^T$, and then again at $\mathbf{q} = (2,1)^T$,

v.

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{pmatrix} xy \\ yz \end{pmatrix}$$

at $\mathbf{q} = (1,-1,2)^T$.

Hint Most of these functions have appeared in previous questions. It may save time to quote the results already proved.

15. Find the Tangent plane to the graph of

$$\phi(\mathbf{x}) = \frac{x^3 - y^3 + 1}{(x+y)^4 + 1},$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, at the point $(2, -1, 5)^T$ *on* the graph.

Hint Multiply up before you differentiate.

16 Find the Tangent plane to the graph of

$$\phi(\mathbf{x}) = \frac{x^2 y + 2xy^2}{1 + x^2 + y^2}$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, at the point $\mathbf{q} = (1, 2)^T$.